On the Threshold-Width of Graphs

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Abstract. The $\mathcal{G}$-width of a class of graphs $\mathcal{G}$ is defined as follows. A graph G has $\mathcal{G}$-width k if there are k independent sets $\mathbb{N}_1, \ldots, \mathbb{N}_k$ in G such that G can be embedded into a graph $H \in \mathcal{G}$ such that for every edge e in H which is not an edge in G, there exists an i such that both endpoints of e are in $\mathbb{N}_i$. For the class $\mathfrak{T}\mathfrak{H}$ of threshold graphs we show that $\mathfrak{T}\mathfrak{H}$-width is NP-complete and we present fixed-parameter algorithms. We also show that for each k, graphs of $\mathfrak{T}\mathfrak{H}$-width at most k are characterized by a finite collection of forbidden induced subgraphs.

1 Introduction

Definition 1. *Let $\mathcal{G}$ be a class of graphs which contains all cliques. The* $\mathcal{G}$-width *of a graph G is the minimum number k of independent sets $\mathbb{N}_1, \ldots, \mathbb{N}_k$ in G such that there exists an embedding $H \in \mathcal{G}$ of G such that for every edge $e = (x, y)$ in H which is not an edge of G there exists an i with $x, y \in \mathbb{N}_i$.*

We restrict the $\mathcal{G}$-width parameter to classes of graphs that contain all cliques to ensure that it is well-defined for every (finite) graph.

In this paper we investigate the width-parameter for the class $\mathfrak{T}\mathfrak{H}$ of threshold graphs and henceforth we call it the *threshold-width* or $\mathfrak{T}\mathfrak{H}$*-width*. If a graph G has threshold-width k then we call G also a k*-probe threshold graph*. We refer to the *partitioned case* of the problem when the collection of independent sets $\mathbb{N}_i$, $i = 1, \ldots, k$, which are not necessarily disjoint, is a part of the input. A collection of independent sets $\mathbb{N}_i$, $i = 1, \ldots, k$, is a *witness* for a partitioned graph. For historical reasons we call the set of vertices $\mathbb{P} = V - \bigcup_{i=1}^{k} \mathbb{N}_i$ the set of *probes* and the vertices of $\bigcup_{i=1}^{k} \mathbb{N}_i$ the set of *nonprobes*.

Threshold graphs were introduced in [3] using a concept called 'threshold dimension.' There is a lot of information about threshold graphs in the book [15], and there are chapters on threshold graphs in the book [6] and in the survey [1].

We may take the following characterization as a definition [3, 1].

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Definition 2. *A graph* G *is a* threshold graph *if* G *and its complement* $\bar{G}$ *are trivially perfect. Equivalently,* G *is a threshold graphs if* G *has no induced* P_4*,* C_4*, nor* $2K_2$*.*

We end this section with some notational conventions. For two sets A and B we write $A + B$ and $A - B$ instead of $A \cup B$ and $A \setminus B$. We write $A \subseteq B$ if A is a subset of B with possible equality and we write $A \subset B$ if A is a subset of B and $A \neq B$. For a set A and an element x we write $A + x$ instead of $A + \{x\}$ and $A - x$ instead of $A - \{x\}$. In those cases we will make it clear in the context that x is an element and not a set.

A graph G is a pair $G = (V, E)$ where the elements of V are called the vertices of G and where E is a set of two-element subsets of V, called the edges. We denote edges of a graph as (x, y) and we call x and y the endvertices of the edge. For a vertex x we write $N(x)$ for its set of neighbors and for $W \subseteq V$ we write $N(W) = \bigcup_{x \in W} N(x) - W$ for the neighbors of W. We write $N[x] = N(x) + x$ for the closed neighborhood of x. For a subset W we write $N[W] = N(W) + W$. Usually we will use $n = |V|$ to denote the number of vertices of G and we will use $m = |E|$ to denote the number of edges of G.

For a graph $G = (V, E)$ and a subset $S \subseteq V$ of vertices we write $G[S]$ for the subgraph *induced* by S, that is the graph with S as its set of vertices and with those edges of E that have both endvertices in S. For a subset $W \subseteq V$ we will write $G - W$ for the graph $G[V - W]$ and for a vertex x we will write $G - x$ rather than $G - \{x\}$. We will usually denote graph classes by calligraphic capitals.

In the next section we show that the class of graphs with $\mathfrak{TH}$-width at most k is characterized by a finite collection of forbidden induced subgraphs.

2 A finite characterization

A graph is a threshold graph if and only if it has no induced P_4, C_4, or $2K_2$. We show that for any k, the class of graphs with $\mathfrak{TH}$-width at most k is characterized by a finite collection of forbidden induced subgraphs.

Lemma 1. *A graph* G *is a threshold graph if and only if it has a binary tree-decomposition* (T, f)*, where* f *is a bijection from the vertices of* G *to the leaves of* T*. Every internal node of* T*, including the root is labeled either as a join– or a union-node. For every internal node the right subtree consists of a single leaf. Two vertices are adjacent in* G *if their lowest common ancestor in* T *is a join-node.*

Proof. According to Theorem 2 on the facing page a graph is a threshold graph if and only if every induced subgraph has either a isolated vertex or a universal vertex. This proves the lemma. □

Theorem 1. *For every* k *the class of graphs with* $\mathfrak{TH}$*-width at most* k *is characterized by a finite collection of forbidden induced subgraphs.*

Proof. To prove this theorem we use the technique introduced by Pouzet [23].

Obviously, the class of graphs with $\mathfrak{Th}$-width at most k is hereditary. Let k be fixed. Assume that the class of $\mathfrak{Th}$-width at most k has an infinite collection of minimal forbidden induced subgraphs, say $G_1, G_2, \ldots$. In each G_i single out one vertex r_i and let $G'_i = G_i - r_i$. Then G'_i has $\mathfrak{Th}$-width at most k, thus there are independent sets $\mathbb{N}_1^{(i)}, \ldots, \mathbb{N}_k^{(i)}$ in G'_i such that G'_i can be embedded into a threshold graph H_i by adding certain edges between vertices that are pairwise contained in some $\mathbb{N}_\ell^{(i)}$. For each i consider a binary tree-decomposition (T_i, f_i) for H_i as stipulated in Lemma 1. Each leaf is labeled by a 0/1-vector with k entries. The j^{th} entry of this vector is equal to 0 or 1 according to whether the vertex is contained in $\mathbb{N}_j^{(i)}$ or not. Thus two vertices are adjacent in G'_i if and only if their lowest common ancestor is a join-node and their vectors are disjoint.

We give each leaf an additional 0/1-label that indicates whether the vertex that is mapped to that leaf is adjacent to r_i or not.

Kruskal's theorem [14] states that binary trees, with points labeled by a well-quasi-ordering are well-quasi-ordered with respect to their lowest common ancestor embedding. When we apply Kruskal's theorem to the labeled binary trees T_i that represent the graphs G'_i we may conclude that there exist $i < j$ such that G'_i is an induced subgraph of G'_j. But then we must also have that G_i is an induced subgraph of G_j. This is a contradiction because we assume that the graphs G_i are minimal forbidden induced subgraphs. This proves the theorem. □

3 $\mathfrak{Th}$-width is fixed-parameter tractable

In this section we show that for constant k, k-probe threshold graphs can be recognized in $O(n^3)$ time.

The following is one of the many characterizations of threshold graphs.

Theorem 2 ([4, 8, 16, 19]). *A graph is a threshold graph if and only if every induced subgraph has an isolated vertex or a universal vertex.*[3]

The following theorem is a monadic second-order characterization. Problems that can be formulated in monadic second-order logic can be solved on graphs that have bounded rankwidth [5]. We will show that k-probe threshold graphs have bounded rankwidth shortly.

Theorem 3. *A graph* $G = (V, E)$ *has threshold-width at most* k *if and only if there exist* k *independent sets* $\mathbb{N}_i$, $i = 1, \ldots, k$, *such that for every* $W \subseteq V$, $G[W]$ *has an isolated vertex or a vertex* ω *such that for every* $y \in W - \omega$ *either* ω *is adjacent to* y *or there exists* $i \in \{1, \ldots, k\}$ *with* $\{\omega, y\} \subseteq \mathbb{N}_i$.

Proof. This is inferred by Theorem 2 and Definition 1. □

[3] A vertex x of a graph G is *isolated* if its neighborhood is the empty set. A vertex x of a graph $G = (V, E)$ is *universal* if $N[x] = V$.

Definition 3 ([22, 20]). *A* rank-decomposition *of a graph* $G = (V, E)$ *is a pair* (T, τ) *where* T *is a ternary tree and* τ *a bijection from the leaves of* T *to the vertices of* G*. Let* e *be an edge in* T *and consider the two sets* A *and* B *of leaves of the two subtrees of* $T - e$*. Let* M_e *be the submatrix of the adjacency matrix of* G *with rows indexed by the vertices of* A *and columns indexed by the vertices of* B*. The* width of e *is the rank over* $GF(2)$ *of* M_e*. The* width of (T, τ) *is the maximum width over all edges* e *in* T *and the* rankwidth *of* G *is the minimum width over all rank-decompositions of* G*.*

Computing the rankwidth of a graph is NP-complete [9] but it is fixed-parameter tractable. This can be seen in various ways: [21] Proves that there is a finite obstruction set for fixed-parameter rankwidth. Now, note that Schrijver describes a general algorithm to minimize a class of submodular functions which uses the ellipsoid method [24, Chapter 45]. He turns this into a 'combinatorial algorithm' for a seemingly larger class of submodular functions, in [25]. Using this result, [9] describes a combinatorial fixed-parameter algorithm for computing the rankwidth of matroids.

Lemma 2. *Threshold graphs have rankwidth at most one.*

Proof. The class of graphs with rankwidth at most 1 is exactly the class of distance-hereditary graphs [20]. Every threshold graph is distance hereditary (see, *e.g.*, [1, 8, 16]). □

Theorem 4. k*-Probe threshold graphs have rankwidth at most* 2^k*.*

Proof. Consider a rank-decomposition (T, τ) with width 1 for an embedding H of G. Consider an edge e in T and assume that M_e is an all-1s-matrix. Each independent set $\mathbb{N}_i$ creates a 0-submatrix in M_e. If $k = 1$ this proves that the rankwidth of G is at most 2. In general, for $k \geqslant 0$, note that there are at most 2^k different neighborhoods from one leaf-set of $T - e$ to the other. It follows that the rank of M_e is at most 2^k. By the way, it is easy to see that this matrix has indeed rank 2^k in the worst case. □

Theorem 5. *For each* $k \geqslant 0$ *there exists an* $O(n^3)$ *algorithm which checks whether a graph* G *with* n *vertices is a* k*-probe threshold graph. Thus* $\mathfrak{T}\mathfrak{H}$*-width* $\in$ FPT.

Proof. k-Probe threshold graphs have bounded rankwidth. C_2MS-Problems can be solved in $O(n^3)$ time for graphs of bounded rankwidth [5, 10, 20]. By Theorem 3, the recognition of k-probe threshold graphs is such a problem.

Alternatively, the theorem is also proved by using the finite collection of forbidden induced subgraphs. Note however that this proof is non-constructive; Kruskal's theorem does not provide the forbidden induced subgraphs. □

A fortiori, Theorem 5 holds as well when the collection of independent sets $\mathbb{N}_1, \ldots, \mathbb{N}_k$ is a part of the input. Thus for each k there is an $O(n^3)$ algorithm that checks whether a graph G, given with k independent sets $\mathbb{N}_i$, can be embedded into a threshold graph.

There are a few drawbacks to this solution. First of all, Theorem 5 only shows the *existence* of an $O(n^3)$ recognition algorithm; *a priori*, it is unclear how to obtain an algorithm explicitly. Furthermore, the constants involved in the algorithm make the solution impractical; already there is an exponential blow-up when one moves from threshold-width to rankwidth.

In the next section we show that there exists an explicit, linear-time algorithm for the recognition of partitioned k-probe threshold graphs.

4 Recognition of partitioned k-probe threshold graphs

In this section, let $(G, \mathcal{N})$ be a partitioned k-probe threshold graph, consisting of a graph G and a k-witness $\mathcal{N}$.

Lemma 3. *If G has an isolated vertex x then G is partitioned k-probe threshold if and only if $G - x$ is partitioned k-probe threshold with the induced collection of independent sets. The same statement holds as well for the unpartitioned case.*

Proof. Assume G is k-probe threshold. Consider an embedding H of G. Then $H - x$ is an embedding of $G - x$. Thus $G - x$ is k-probe threshold.
Assume $G - x$ is k-probe threshold. Let H' be an embedding of $G - x$. Then we obtain an embedding of G by adding x as an isolated vertex to H'. □

Theorem 6. *For every k there exists a linear-time algorithm to check whether a pair $(G, \mathcal{N})$, where G is a graph and $\mathcal{N}$ a collection of k independent sets in G, is a partitioned k-probe threshold graph.*

Proof. Assume that $(G, \mathcal{N})$ is a partitioned graph and let H be an embedding of G. If H has an isolated vertex x, then x is also isolated in G since H is an embedding of G. By Lemma 3 any isolated vertex may be safely removed from G.
Now we may assume that any embedding H is connected. By Theorem 2 H has a universal vertex ω. We call ω a 'probe universal vertex' of $(G, \mathcal{N})$ if for every nonneighbor z there is an independent set in $\mathcal{N}$ which contains both ω and z. Thus any partitioned k-probe threshold graph has an isolated vertex or a probe universal vertex. Finally, observe the following: if ω is a probe universal vertex then G is k-probe threshold if and only if $G - \omega$ is k-probe threshold, since we may add ω as a universal vertex to any embedding of $G - \omega$ and obtain an embedding of G. Since k is a constant, an elimination ordering by isolated - and probe universal vertices can be obtained in linear time. □

Remark 1. Note that the algorithm described in Theorem 6 is fully polynomial. This proves that the 'sandwich problem,' studied by Golumbic *et al.*, in [7], is polynomial for threshold graphs.

5 $\mathfrak{TH}$-width is NP-complete

Let $\mathfrak{T}$ be the class of complete graphs (cliques). We proved in [2] that $\mathfrak{T}$-width is NP-complete. For completeness sake we include the proof.

Theorem 7. *$\mathfrak{T}$-Width is NP-complete.*

Proof. Let $(G, \mathcal{N})$ be a partitioned k-probe complete graph, with a witness

$$\mathcal{N} = \{\mathbb{N}_i \mid i = 1, \ldots, k\}$$

which is a collection of k independent sets in G. Thus every non-edge of G has both its endvertices in one of the independent sets $\mathbb{N}_i$. Then $\mathcal{N}$ forms a clique-cover of the edges of $\bar{G}$. This shows that a graph G has $\mathfrak{T}$-width at most k if and only if the edges of $\bar{G}$ can be covered with k cliques. The problem to cover the edges of a graph by a minimum number of cliques is NP-complete [13]. This proves the theorem. □

Theorem 8. *$\mathfrak{TH}$-width is NP-complete.*

Proof. Assume there is a polynomial-time algorithm to compute $\mathfrak{TH}$-width. We show that we can use that algorithm to compute $\mathfrak{T}$-width. Let G be a graph for which we wish to compute $\mathfrak{T}$-width. Construct a graph G' by adding a clique C with n^2 vertices. Make all vertices of C adjacent to all vertices of G. Add one more vertex ω and make ω also adjacent to all vertices of G. Consider two nonadjacent vertices x and y of G. In any embedding of G' into a threshold graph, either x and y are adjacent or ω is adjacent to all vertices of C. However, to make ω adjacent to all vertices of C, we need at least n^2 independent sets. Obviously, making a clique of G embeds G' into a threshold graph, namely the complement of a star and a collection of isolated vertices. This embedding needs less than n^2 independent sets. This proves the theorem. □

6 A fixed-parameter algorithm to compute $\mathfrak{TH}$-width

Assume that $(G, \mathcal{N})$ is a connected partitioned k-probe threshold graph with witness

$$\mathcal{N} = \{\mathbb{N}_i \mid i = 1 \ldots, k\}$$

and let H be an embedding. The *label* $L(x)$ of a vertex x is the 0/1-vector of length k with the i^{th} entry $L_i(x)$ equal to 1 if and only if $x \in \mathbb{N}_i$. We write $L(x) \leqslant L(y)$ if $L_i(x) \leqslant L_i(y)$ for all $i = 1, \ldots, k$. We write $L(x) \perp L(y)$ if there is no i with $L_i(x) = L_i(y) = 1$.

Definition 4. *A witness $\mathcal{N}$ is* well-linked *if for every $i = 1, \ldots, k$, every vertex $x \notin \mathbb{N}_i$ has a neighbor in $\mathbb{N}_i$.*

Lemma 4. *Every k-probe threshold graph has a witness with k independent sets which is well-linked.*

Proof. Starting with any witness, repeatedly add a vertex x to an independent set $\mathbb{N}_i$ if it has no neighbor in that set. $\square$

Consider the equivalence relation $\equiv$ defined by $x \equiv y$ if $N(x) = N(y)$. Denote the equivalence class of a vertex x by (x). Define the partial order $\preceq$ by:

$$(x) \preceq (y) \quad \text{if} \quad N(x) \subseteq N(y).$$

Likewise, we consider the equivalence relation $\equiv'$ defined by $x \equiv' y$ if $N[x] = N[y]$. The equivalence class of a vertex x under this relation is denoted by $[x]$. We consider the partial order defined by:

$$[x] \preceq [y] \quad \text{if} \quad N[x] \subseteq N[y].$$

Lemma 5. *Assume* $(G, \mathcal{N})$ *is a* k*-probe clique with a well-linked witness* $\mathcal{N}$*. Then*

$$(x) \preceq (y) \quad \Leftrightarrow \quad L(x) \geqslant L(y) \neq \mathbf{0}.$$

Proof. Assume $(x) \preceq (y)$. Thus $N(x) \subseteq N(y)$. Assume that $y \in \mathbb{N}_i$ and $x \notin \mathbb{N}_i$ for some i. Since $\mathcal{N}$ is well-linked there exists a vertex $z \in N(x) \cap \mathbb{N}_i$. Then $z \in N(y)$ since $N(x) \subseteq N(y)$. But this contradicts $\{y, z\} \subseteq \mathbb{N}_i$.
Assume $L(x) \geqslant L(y) \neq \mathbf{0}$. Then x and y are not adjacent. Let $z \in N(x)$. Then $L(z) \perp L(x)$. Since $L(x) \geqslant L(y)$ also $L(z) \perp L(y)$. Thus $z \in N(y)$ since $(G, \mathcal{N})$ is a k-probe clique. $\square$

Note that Definition 2 is equivalent to the following characterization.

Theorem 9 ([4, 15]). *A graph* H *is a threshold graph if and only if for every pair of vertices* x *and* y*,* $N(x) \subseteq N[y]$ *or* $N(y) \subseteq N[x]$*.*

In other words, a graph $G = (V, E)$ is a threshold graph if and only if there is a *total order* of the vertices $[x_1, \ldots, x_n]$, *i.e.*, a *chain*, such that:

$$1 \leqslant i < j \leqslant n \quad \Rightarrow \quad N(x_i) \subseteq N[x_j].$$

Theorem 10. *Let* $(G, \mathcal{N})$ *be a* k*-probe threshold graph with a well-linked witness* $\mathcal{N}$ *and let* H *be an embedding. For every nonadjacent pair* x *and* y *in* G *with* $N_H(x) \subseteq N_H[y]$*:*

$$(x) \preceq (y) \quad \Leftrightarrow \quad L(x) \geqslant L(y).$$

Proof. Assume $L(x) \geqslant L(y)$. Let $z \in N_G(x)$. Then $z \in N_H[y]$. Since x and y are not adjacent, $z \neq y$. Thus $z \in N_H(y)$. If $z \notin N_G(y)$, then there exists an i with $\{z, y\} \subseteq \mathbb{N}_i$. Now $L(x) \geqslant L(y)$ implies that also $x \in \mathbb{N}_i$, which contradicts that z is adjacent to x. Hence $(x) \preceq (y)$.
Assume $(x) \preceq (y)$, that is, $N_G(x) \subseteq N_G(y)$. *A fortiori*, x and y are not adjacent. Assume $\neg(L(x) \geqslant L(y))$. Then there exists an i with $y \in \mathbb{N}_i$ and $x \notin \mathbb{N}_i$. Since $\mathcal{N}$ is well-linked, there exists a vertex $z \in N_G(x) \cap \mathbb{N}_i$. Since $(x) \preceq (y)$, $z \in N_G(y)$, contradicting that z and y are both in $\mathbb{N}_i$. $\square$

For completeness sake we note the following.

Lemma 6. *Assume* $[x] \preceq [y]$ *and* $x \neq y$. *Then*

$$\forall_{i \in L(y)} N_G(x) \cap \mathbb{N}_i = \{y\}.$$

Proof. Since x and y are adjacent, we have that $L(x) \perp L(y)$. Assume that $y \in \mathbb{N}_i$ for some $i \in \{1, \ldots, k\}$. Thus $x \notin \mathbb{N}_i$. Since $\mathcal{N}$ is well-linked, there exists a vertex $z \in N(x) \cap \mathbb{N}_i$. Since $N_G[x] \subseteq N_G[y]$, $z \in N_G[y]$. But then we must have $z = y$, otherwise z and y are nonadjacent. □

Definition 5. *A* true – *or* false module *is a set of vertices such that every pair is a true – or false twin, respectively.*[4] *A* k-probe module *is either a false module with at least 3 vertices or a true module with at least* $k + 3$ *vertices.*

Lemma 7. *Let* S *be* k*-probe module. Then* G *has* $\mathfrak{TH}$*-width at most* k *if and only if* $G - x$ *has* $\mathfrak{TH}$*-width at most* k *for any* $x \in S$.

Proof. If G is k-probe threshold then so is $G - x$ for any vertex x. Let $x \in S$ and assume that $G - x$ is a k-probe threshold graph. Let H be an embedding of $G - x$. First assume that S is a false module with at least three vertices. Let $y \in S - x$. If y is in the independent set, then we can let x be a copy of y. Assume that all vertices of $S - x$ are in the clique of H. Since $S - x$ has at least two vertices, they must be nonprobes. We can let x be a copy of either of them.
Assume S is a true module with at least $k + 3$ vertices. Then at least $k + 1$ vertices are in the clique C of H. Let z be a vertex of $S \cap C$ with a minimal closed neighborhood in H. Assume that z has a neighbor u in H which is not a neighbor of z in G. Then u is a neighbor of every vertex of $S \cap C$ in H, but not in G. Since every pair of vertices $a, b \in S$ is adjacent in G, $L(a) \perp L(b)$. It follows that u must be in at least $k + 1$ independent sets, which is a contradiction. Thus $N_H(z) = N_G(z)$, and we can let x be a copy of z. □

Definition 6. *A vertex* x *is* maximal *if there exists no* $(y) \neq (x)$ *with* $(x) \preceq (y)$ *and there exists no* $[y] \neq [x]$ *with* $[x] \preceq [y]$.

Lemma 8. *Assume that* G *is a* k*-probe threshold graph without* k*-probe module. Then there are at most* $2^{k+1} + k$ *maximal vertices.*

Proof. Consider a well-linked embedding H. By Theorem 9 there is a chain order of its vertices. Let $M_0, M_1, \ldots$ be the equivalence classes in H of vertices with the same open or closed neighborhoods. Assume they are ordered such that $N[x_i] \supseteq N(x_{i+1})$ for each $x_i \in M_i$ and $x_{i+1} \in M_{i+1}$, for $i = 0, 1, \ldots$. Thus if H is connected, M_0 is the set of universal vertices in H. We call these equivalence classes $M_0, M_1, \ldots$ the *levels* of the embedding. Thus a level contained in the clique induces a k-probe clique in G and a level contained in the independent set induces an independent set in G.

[4] A *true twin* is a pair of vertices x and y with $N[x] = N[y]$. A *false twin* is a pair of vertices x and y with $N(x) = N(y)$.

Consider the partition of each level M_s into sets of vertices with the same label. We call the sets of the partition of a level M_s the *label-sets* of M_s. Notice that each label-set is a module in G. Since there is no k-probe module, each label-set of nonprobes has at most 2 vertices and the label-set of probes has at most $k+2$ vertices. Thus

$$|M_s| \leqslant 2(2^k - 1) + (k+2) = 2^{k+1} + k.$$

By Theorem 10 a vertex $x \in M_s$ is maximal if it has a label $L(x)$ such that all other label-sets $L' \leqslant L(x)$ in $M_0, \ldots, M_s$ are empty. It follows that there are at most $\sum_{i=0}^{k} \binom{k}{i} = 2^k$ label-sets of maximal vertices, at most $2^k - 1$ of maximal nonprobes, each containing at most 2 elements, and at most one label-set of maximal probes, containing at most $k+2$ elements. Thus the number of maximal elements is bounded by $2^{k+1} + k$. □

Lemma 9. *Assume G is a k-probe threshold graph without isolated vertices and without k-probe module. There exists a set Υ, of size $|\Upsilon| \leqslant 2^{2(k+1)}$ such that any well-linked embedding of G has its set of universal vertices $M_0 \subseteq \Upsilon$. This set Υ can be computed in linear time.*

Proof. Since G has no isolated vertices, H has a set of universal vertices M_0. Start with $\Upsilon = \varnothing$. Repeatedly compute the set of maximal vertices in G, add them to Υ, and delete them from the graph. After at most 2^k repetitions, each label-set of M_0 is contained in Υ. Since each set of maximal elements has at most $2^{k+1} + k$ vertices,

$$|\Upsilon| \leqslant 2^k(2^{k+1} + k) \leqslant 2^{2k+1} + 2^{2k} \leqslant 2^{2(k+1)}.$$

□

Definition 7. *A* probe universal set *is a set U of labeled vertices such that for every vertex $x \notin U$, there is a label for x such that $U + x$ is a partitioned probe clique.*

Lemma 10. *Let U be a probe universal set and let $x \notin U$ be a vertex with minimal neighborhood such that $U' = U + N(x)$ is probe universal with the same number of nonempty label-sets as U. Then there exists an embedding such that U is universal if and only if there exists an embedding such that U' is universal.*

Proof. By definition, the label-sets of U' are modules that extend the label-sets of U. This proves the lemma. □

Theorem 11. *For each k, there exists an $O(n^2)$-time algorithm for the recognition of k-probe threshold graphs.*

Proof. We may assume that G has no k-probe module. By Lemma 9 there exists a constant number of feasible universal sets. By Lemma 10, if there exists a vertex x that can be labeled such that $N(x)$ extends the universal set in a way that does not increase the number of nonempty label-sets in the universal set, then the algorithm can greedily extend the universal set with $N(x)$. Next the algorithm removes the vertex x and tries to find another greedy extension.

If there are no more greedy extensions, the algorithm computes the set Υ as in Lemma 9 in the graph minus the probe universal set, and chooses one of the constant number of subsets as an extension of the probe universal set. Notice that there can be at most 2^k extensions that increase the number of label-sets.

Since the computation of maximal vertices can be done in $O(n^2)$ time, the algorithm can be implemented to run in $O(n^2)$ time. □

Remark 2. Perhaps it is a bit surprising that we don't have to treat the different components of the graph separately.

7 Concluding remarks

The recognition problem of probe interval graphs was introduced by Zhang *et al.* [26, 17]. This problem stems from the physical mapping of chromosomal DNA of humans and other species. Since then probe graphs of many other graph classes have been investigated by various authors. In this paper we generalized the concept to the graph-class-width parameters. So far, we have limited our research to classes of graphs that have bounded rankwidth.

In [11], we derived a fixed-parameter algorithm that solves a similar problem for the class of trivially perfect graphs. It is well-known that every threshold graph is trivially perfect. Obviously, this does not imply that the algorithm for trivially perfect graphs can be used for threshold graphs. In fact, a similar, elegant solution as the one that we obtained in this paper can not work for threshold graphs.

For the classes of blockgraphs, threshold graphs, trivially perfect graphs, and cographs we were able to show that the width parameter is fixed-parameter tractable [2, 11, 12]. One of the classes for which this is still open is the class of distance-hereditary graphs. We are unaware of a monadic second-order formulation that describes the distance-hereditary width. Consider a decomposition tree of bounded rankwidth. The 'twinset' of a branch is defined as the subset of vertices that are mapped to the leaves of that branch, and that have neighbors in the rest of the graph (outside the branch). It is not difficult to show that for bounded rankwidth, the graphs that arise as twinsets constitute a class of graphs that is characterized by a finite collection of forbidden induced subgraphs. (For rankwidth one this is the class of cographs.) The same holds true for graphs of bounded DH-width. So far, we have not been able to describe the class of graphs as tree-extensions of these twinsets.

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